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Approximating J -holomorphic curves by holomorphic ones

Received: 2 October 2003 / Accepted: 18 November 2003

Published online: 25 February 2004 – © Springer-Verlag 2004

Abstract. Given an almost complex structure J in a cylinder of \mathbb{R}^{2p} ($p > 1$) together with a compatible symplectic form ω and given an arbitrary J -holomorphic curve Σ without boundary in that cylinder, we construct an holomorphic perturbation of Σ , for the canonical complex structure J_0 of \mathbb{R}^{2p} , such that the distance between these two curves in $W^{1,2}$ and L^∞ norms, in a sub-cylinder, are controled by quantities depending on J, ω and by the area of Σ only. These estimates depend neither on the topology nor on the conformal class of Σ . They are key tools in the recent proof of the regularity of 1-1 integral currents in [RT].

1 Introduction

Let ω be a smooth symplectic form in $B_2^2(0) \times B_2^{2p-2}(0)$ ($p > 1$) - ω is a closed 2-form satisfying $\omega^p > 0$ - and let J be a smooth compatible almost complex structure : $g(\cdot, \cdot) := \omega(\cdot, J\cdot)$ is symmetric and therefore defines a scalar product in $B_2^2(0) \times B_2^{2p-2}(0)$ that we will denote by g . We assume that at the origin $\omega(0)$ coincides with the standard symplectic form of \mathbb{R}^{2p} , $\omega_0 = \sum_{i=1}^p dx_{2i-1} \wedge dx_{2i}$ and that $J(0)$ coincides with the standard almost-complex structure J_0 satisfying $J_0 \cdot e_{2i-1} = e_{2i}$ for $i = 1 \cdots p$ where e_k is the canonical basis of \mathbb{R}^{2p} .

We consider a J holomorphic curve $\Psi : \Sigma \rightarrow B_2^2(0) \times B_2^{2p-2}(0)$ (Σ is a smooth Riemann surface and Ψ a smooth J -holomorphic map from Σ into $(B_2^2(0) \times B_2^{2p-2}(0), J)$). we assume that the current $\Psi_*[\Sigma]$ satisfies

$$\text{supp}(\partial(\Psi_*[\Sigma])) \subset \partial B_2^2(0) \times B_2^{2p-2}(0) \quad . \quad (1.1)$$

We will adopt the following notation : for any $r < 2$

$$\Sigma_r := \Psi^{-1}(B_r^2(0) \times B_2^{2p-2}(0)) \quad . \quad (1.2)$$

(Under these notations one has for instance $\Sigma_2 = \Sigma$). We define now the “distortions” of $g(\cdot, J\cdot)$ relative to the canonical flat metric g_0 in $B_2^2(0) \times B_2^{2p-2}(0)$.

These are the following quantities :

$$\begin{aligned} d_1(g) &:= \sup_x \sup_{X \in T_x \mathbb{R}^{2p}} \frac{g(X, X)}{g_0(X, X)} + \frac{g_0(X, X)}{g(X, X)} \\ d_2(g) &:= \sup_x \sup_{X, Y \in T_x \mathbb{R}^{2p}} \frac{g(X \wedge Y, X \wedge Y)}{g_0(X \wedge Y, X \wedge Y)} + \frac{g_0(X \wedge Y, X \wedge Y)}{g(X \wedge Y, X \wedge Y)} . \end{aligned} \quad (1.3)$$

Our main result in this paper is the following.

Theorem 1.1 *For any J holomorphic curve $\Psi : \Sigma \rightarrow B_2^2(0) \times B_2^{2p-2}(0)$ satisfying (1.1), there exists a map $\eta : \Sigma \rightarrow \mathbb{R}^{2p}$, such that $\Psi + \eta$ is J_0 holomorphic and η satisfies*

$$\int_{\Sigma} |\nabla \eta|^2 \leq 2 \|J - J_0\|_{\infty}^2 \int_{\Sigma} \Psi^* \omega , \quad (1.4)$$

and

$$\|\eta\|_{L^{\infty}(\Sigma_1)} \leq K \|\nabla J\|_{\infty} , \quad (1.5)$$

where K is a constant depending only on $d_1(g)$, $d_2(g)$ and $\int_{\Sigma} \Psi^* \omega$.

The striking fact in this result is that the constants are independent of the conformal type and the topology of Σ . These estimates are key tools in the proof of the regularity of 1-1 integral currents in arbitrary dimension in [RT]. η is choised to be the solution of the following elliptic problem – see Proposition A.3 –

$$\left\{ \begin{array}{ll} \bar{\partial} \eta = -\bar{\partial} \Psi & \text{in } \mathcal{D}'(\Sigma) \\ \forall h \in H_0^+(\Sigma) & \int_{\partial \Sigma} \eta \, dh = 0 \\ \forall \Sigma_k \text{ connected compo. of } \Sigma , & \int_{\partial \Sigma_k} \eta = 0 \end{array} \right. . \quad (1.6)$$

where we are representing η and Ψ by their canonical complex coordinates in (\mathbb{R}^{2p}, J_0) and where $H_0^+(\Sigma)$ denotes the space of $W^{1,2}(\Sigma)$ holomorphic functions on Σ . Observe that since Ψ is J -holomorphic, taking the $\bar{\partial}$ of the first equation in (1.6) one gets for all $k = 1 \cdots 2p$ (using the real coordinates this time)

$$\Delta_{\Sigma} \eta^k = - * \left(\sum_{l=1}^{2p} d(J_l^k(\Psi)) \wedge d\Psi^l \right) \quad \text{in } \mathcal{D}'(\Sigma) . \quad (1.7)$$

Since $\int_{\Sigma} |\nabla \psi|^2 = 2 \int_{\Sigma} \Psi^* \omega$ which is one of the variable of the problem one is led to a first order formulation of Wente's Problem : Let u be a function on Σ satisfying

$$\left\{ \begin{array}{ll} \bar{\partial} u = f & \text{in } \mathcal{D}'(\Sigma) \\ \forall h \in H_0^+(\Sigma) & \int_{\partial \Sigma} u \, dh = 0 \\ \forall \Sigma_k \text{ connected compo. of } \Sigma , & \int_{\partial \Sigma_k} u = 0 \end{array} \right. , \quad (1.8)$$

where f is a L^2 $\bar{\partial}$ exact $(0, 1)$ form $f = \bar{\partial}\phi$ satisfying

$$*\partial f = da \wedge db \quad , \quad (1.9)$$

where a and b are $W^{1,2}$ functions in Σ . Assuming f is $L^2(\Sigma)$ perpendicular to $\bar{\partial}H_0^-(\Sigma) \oplus \bar{\partial}V$, where $H_0^-(\Sigma)$ is the space of anti-holomorphic functions in Σ and V is the finite dimensional space of harmonic functions in Σ which are constant on each connected component of $\partial\Sigma$, then one easily verifies, see the appendix, that the harmonic extension \tilde{u} is perpendicular to $H_0^+(\Sigma) \oplus H_0^-(\Sigma) \oplus V$ and therefore is equal to 0. Thus u satisfies

$$\begin{cases} *\Delta u = da \wedge db & \text{in } \mathcal{D}'(\Sigma) \\ u = 0 & \text{on } \partial\Sigma \end{cases} \quad (1.10)$$

and from P.Topping's result [To] one has

$$\|u\|_{L^\infty(\Sigma)} \leq \frac{1}{2\pi} \|\nabla a\|_{L^2(\Sigma)} \|\nabla b\|_{L^2(\Sigma)} \quad , \quad (1.11)$$

(see more on the second order Wentz Problem in [Ge] and [He]). Therefore if one would know that Ψ is perpendicular to $H_0^-(\Sigma) \oplus V$ we would directly have obtained

$$\|\eta\|_{L^\infty(\Sigma)} \leq \frac{4p}{\pi} \int_{\Sigma} \Psi^* \omega \|\nabla J\|_{\infty} \quad . \quad (1.12)$$

Of course there is no reason for Ψ to satisfy this assumption and the difficulty comes then from the L^2 projection of Ψ over $H_0^-(\Sigma) \oplus V$. A solution η to the problem (1.8) and (1.9) can even not be bounded in L^∞ on the whole Σ . Take for instance $\Sigma = D^2$ and

$$f = \bar{\partial} \left(\sum_{n=1}^{\infty} \frac{1}{n \log n} e^{-i n \theta} \right) \quad .$$

Therefore there is a real need to restrict to a subdomain of Σ as we do in (1.5). In this sense our result is optimal.

In the proof below we were influenced by the proofs in [Ch] and [To] .

2 Proof of Theorem 1.1.

Before to prove Theorem 1.1 we need an intermediate result.

Lemma 2.1 *Let $\Psi : \Sigma \rightarrow B_2^{2p-2}(0) \times B_2^{2p-2}(0)$ be a J -holomorphic curve satisfying (1.1). For any smooth function u whose average on each connected component of Σ is zero, or any function u in $C_0^\infty(\Sigma_2)$, the following inequality holds*

$$\left(\int_{\Sigma_1} |u|^2 \right)^{\frac{1}{2}} \leq K \int_{\Sigma} |\nabla u| \quad , \quad (2.13)$$

where Σ_1 is defined in (1.2), the metric on Σ is the pull-back by Ψ of $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ and K is a constant depending only on $d_1(g)$, $d_2(g)$ and $\int \Psi^* \omega$.

Proof of Lemma 2.1. We present the proof in the case where the average of u vanishes on each connected component of Σ (the other case $u \in C_0^\infty(\Sigma)$ being more easy). Let Σ^k be a connected component of Σ intersecting Σ_1 . We divide Σ^k into 2 subsurfaces $\Sigma^{k,+}$ (res. $\Sigma^{k,-}$) being the subset of Σ^k where u is positive (resp. negative). Using the coarea formula (see [Fe]) we have

$$\int_{\Sigma^{k,+}} |\nabla u| = \int_0^{+\infty} \mathcal{H}^1(u^{-1}(s) \cap \Sigma^k) ds \quad (2.14)$$

Since u is smooth and Σ^k is connected and since 0 is a value of u in Σ^k , for every regular value $s \in \mathbb{R}^+$ of u 2 cases may happen.

Case 1:

$$\exists r \in [1, 2] \quad \text{such that} \quad \partial\{x \in \Sigma_r^k ; u(x) \geq s\} = u^{-1}(s) \cap \Sigma_r^k .$$

In that case, since Σ_r is an area minimizing surface in \mathbb{R}^{2p} for the metric $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$, we have

$$\begin{aligned} (\mathcal{H}^1(u^{-1}(s) \cap \Sigma_r^k))^2 &\geq K_0^{-1} \mathcal{H}^2(u^{-1}([s, +\infty)) \cap \Sigma_r^k) \\ &\geq K_0^{-1} \mathcal{H}^2(u^{-1}([s, +\infty)) \cap \Sigma_1^k) . \end{aligned} \quad (2.15)$$

where K_0 is the isoperimetric constant of (\mathbb{R}^{2p}, g) .

Case 2:

$$\forall r \in [1, 2] \quad \partial\{x \in \Sigma_r^k ; u(x) \geq s\} \neq u^{-1}(s) \cap \Sigma_r^k .$$

This means that, in such a case, $\forall r \in [1, 2] \quad u^{-1}(s) \cap \partial\Sigma_r^k \neq \emptyset$. Since the distance for g in \mathbb{R}^{2p} between $\partial\Sigma_1^k$ and $\partial\Sigma^k$ is larger than $K_1 > 0$, where K_1 only depends on g , we get

$$\mathcal{H}^1(u^{-1}(s) \cap \Sigma_2^k) \geq K_1 . \quad (2.16)$$

Let us denote $K_2 = \int_\Sigma \Psi^* \omega = \mathcal{H}^2(\Sigma)$, where the Hausdorff distance in Σ is computed according to the pull-back metric by Ψ of $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$. We then have in that case

$$\begin{aligned} (\mathcal{H}^1(u^{-1}(s) \cap \Sigma_2^k))^2 &\geq K_1^2 \geq K_1^2 K_2^{-1} \mathcal{H}^2(\Sigma) \\ &\geq K_1^2 K_2^{-1} \mathcal{H}^2(u^{-1}([s, +\infty)) \cap \Sigma_1^k) . \end{aligned} \quad (2.17)$$

Combining (2.14), (2.15) and (2.17), we obtain the existence of K depending only of $d_1(g)$, $d_2(g)$ and $\int_\Sigma \Psi^* \omega$ such that

$$\int_{\Sigma^{k,+}} |\nabla u| \geq K^{-1} \int_0^{+\infty} ds [\mathcal{H}^2(u^{-1}([s, +\infty)) \cap \Sigma_1^k)]^{\frac{1}{2}} \quad (2.18)$$

Observe that the right-hand-side of this last inequality is a multiple of the Lorentz $L^{2,1}$ -norm of u in $\Sigma_1^{k,+}$. We claim that the L^2 -weak norm of $u^+ = \max\{u, 0\}$, $L^{2,\infty}$ on Σ_1^k can be bounded by $\|u\|_{L^2(\Sigma_1^{k,+})}$

$$\|u\|_{L^{2,\infty}(\Sigma_1^{k,+})} = \sup_{s \geq 0} s [\mathcal{H}^2(u^{-1}([s, +\infty)) \cap \Sigma_1^k)]^{\frac{1}{2}} \leq K_3 \|u\|_{L^2(\Sigma_1^{k,+})} \quad (2.19)$$

where K_3 only depends on $d_1(g)$, $d_2(g)$ and $\int_{\Sigma} \Psi^* \omega$. Indeed, we consider in Σ^k the pseudo-distance d_g which is given by the distance in $(B_2^2(0) \times B_2^{2p-2}(0), g)$ – Since Σ^k is not necessarily embedded, it may happens that $d_g(x, y) = 0$ and $x \neq y$. For this pseudo-distance in Σ^k , we consider the balls $B_r^{d_g}(x) := \Psi^{-1}(B_r^{2p}(x)) \cap \Sigma^k$. Since the current $\Psi_*[\Sigma_1^k]$ is area minimizing in $(B_2^2(0) \times B_2^{2p-2}(0), g)$ – it is calibrated by ω -, using the monotonicity formula, we obtain that for every $x \in \Sigma_1^k$ and $r < 1/2$

$$\pi r^2 \leq \mathcal{H}^2(B_r^{d_g}(x)) \leq r^2 \int_{\Sigma} \Psi^* \omega = r^2 K_2$$

Therefore these balls satisfy the doubling property

$$4\mathcal{H}^2(B_r^{d_g}(x))\pi^{-1}K_2 \geq \mathcal{H}^2(B_{2r}^{d_g}(x))$$

We then adapt to our case the proof of the covering lemma page 9 of [St] for m being the 2 Hausdorff measure restricted to Σ_2^k and the balls being balls for the pseudo-distance d_g to get the corresponding statement to that lemma. We can now obtain (2.19) by following the first part of the proof of Theorem 1 page 5 of [St], taking for the covering of pseudo-balls $B_j^{d_g}$ given by the covering lemma but considering this time the metric Ψ^*g on Σ^k . From (2.19) we deduce

$$\begin{aligned} \|u\|_{L^2(\Sigma_1^{k,+})} \int_0^{+\infty} ds [\mathcal{H}^2(u^{-1}([s, +\infty)) \cap \Sigma_1^k)]^{\frac{1}{2}} &\geq \\ K_3^{-1} \int_0^{+\infty} s ds [\mathcal{H}^2(u^{-1}([s, +\infty)) \cap \Sigma_1^k)] &= \|u\|_{L^2(\Sigma_1^{k,+})}^2 \end{aligned} \quad (2.20)$$

Combining now (2.18) and (2.20) we obtain the desired inequality (2.13) for Σ^k instead of Σ . $\Sigma = \cup_k \Sigma^k$. Observing that the number of components Σ^k having some non empty intersection with Σ_1 is bounded by $\int_{\Sigma} \Psi^* \omega$ times a constant depending only of $d_1(g)$, $d_2(g)$ (this is a consequence of the monotonicity formula coming from fact that Σ^k are area minimizing), then we get (2.13) for Σ . this time and Lemma 2.1 is proved. \square

Proof of Theorem 1.1. Using local conformal coordinates ξ_1, ξ_2 in Σ , we have for all $k = 1 \dots 2p$

$$\frac{\partial \Psi^k}{\partial \xi_1} = - \sum_{l=1}^{2p} J_l^k(\Psi) \frac{\partial \Psi^l}{\partial \xi_2} \quad \text{and} \quad \frac{\partial \Psi^k}{\partial \xi_2} = \sum_{l=1}^{2p} J_l^k(\Psi) \frac{\partial \Psi^l}{\partial \xi_1}$$

Taking respectively the ξ_1 derivative and the ξ_2 derivative of these two equations we obtain

$$\forall k = 1 \cdots 2p \quad *(\partial \bar{\partial} \Psi^k) = \Delta_\Sigma \Psi^k = * \left(\sum_{l=1}^{2p} d(J_l^k(\Psi)) \wedge d\Psi^l \right) . \quad (2.21)$$

Since $\Psi + \eta$ is J_0 -holomorphic, using the canonical complex coordinates in \mathbb{R}^{2p} , we have $\bar{\partial}(\Psi + \eta) = 0$ from which we deduce $\Delta_\Sigma(\Psi + \eta) = 0$ and therefore this yields

$$\forall k = 1 \cdots 2p \quad \Delta_\Sigma \eta^k = - * \left(\sum_{l=1}^{2p} d(J_l^k(\Psi)) \wedge d\Psi^l \right) . \quad (2.22)$$

Since Ψ is an isometry for the induced metric, we then deduce from (2.22) that

$$\forall k = 1 \cdots 2p \quad \|\Delta_\Sigma \eta^k\|_{L^\infty(\Sigma)} \leq 4p \|\nabla J\|_\infty \int_\Sigma \Psi^* \omega . \quad (2.23)$$

Let $\chi(t)$ be a smooth cut-off function equal to 1 in $[0, 1]$ and equal to zero for $t \geq 2$ with $\|\chi^l\|_\infty \leq K_l$. We define in $B_2^2(0) \times B_2^{2p-2}(0)$ the cut-off function – that we also denote $\chi - \chi(x) := \chi(x_1^2 + x_2^2)$. For any function ϕ in $B_2^2(0) \times B_2^{2p-2}(0)$ we denote by $\nabla^C \phi$ the tangent vector field to $\Psi(\Sigma)$ obtained by taking the orthogonal projection of the gradient of ϕ for the metric $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$. For any vector field Y in $B_2^2(0) \times B_2^{2p-2}(0)$ we denote by $\text{div}^\Sigma Y$ the divergence along $\Psi(\Sigma)$ of that vector-field (taking normal coordinates (y_1, \dots, y_{2p}) for g in a neighborhood of $x_0 \in \text{supp } \Psi_*[\Sigma]$ we have $\text{div}^\Sigma Y(x_0) = \sum_{l=1}^{2p} \nabla^\Sigma Y_l \cdot \frac{\partial}{\partial y_l}(x_0)$. It is a classical fact that, for a vector field Y normal to $\Psi(\Sigma)$ one has $\text{div}^\Sigma Y = H \cdot Y$ where H is the mean curvature vector of $\Psi(\Sigma)$ which is zero in our case. Therefore we have in particular $\Delta_\Sigma \chi := \text{div}^\Sigma \nabla^\Sigma \chi(x_0) = \text{div}^\Sigma \nabla \chi(x_0) = \sum_{l=1}^{2p} \nabla^\Sigma \left(\frac{\partial \chi}{\partial y_l} \right) \cdot \frac{\partial}{\partial y_l}(x_0)$, still using the normal coordinates in (\mathbb{R}^{2p}, g) about x_0 . Since $|\nabla \frac{\partial \chi}{\partial y_l}|(x) \leq \|\chi\|_{C^2}$, we then deduce that there exists a constant K independent of the variables of our problem such that

$$|\Delta_\Sigma \chi|_\infty \leq K . \quad (2.24)$$

Finally we have, using (2.23) and (2.24)

$$|\Delta_\Sigma(\chi \eta^k)| \leq K \mathbf{1}_{\Sigma_{\sqrt{2}}} \left[|\eta^k| + |\nabla \eta^k| + \|\nabla J\|_\infty \int_\Sigma \Psi^* \omega \right] \quad (2.25)$$

where K only depends on p and where $\mathbf{1}_{\Sigma_{\sqrt{2}}}$ is the characteristic function equal to 1 on $\Sigma_{\sqrt{2}}$ and 0 outside. Using (2.25) and Lemma 2.1 (for $\Sigma_{\sqrt{2}}$ instead of Σ_1) – having chosed η^k with average 0 on each connected component of Σ – we finally have

$$\int_{\Sigma_{\sqrt{2}}} |\Delta_\Sigma(\chi \eta^k)|^2 \leq K \|\nabla J\|_\infty^2 . \quad (2.26)$$

where K only depend on $d_1(g)$, $d_2(g)$ and $\int_{\Sigma} \Psi^* \omega$.

We denote by G_a the Green Function of Δ_{Σ} on $\Sigma_{\sqrt{2}}$ for the zero boundary condition on $\partial \Sigma_{\sqrt{2}}$ (recall that each connected component of $\Sigma_{\sqrt{2}}$ has a boundary since it is an area minimizing surface and therefore posses a Green function – see [FK]). Precisely G_a solves

$$\begin{cases} \Delta_{\Sigma} G_a = \delta_a & \text{in } \Sigma_{\sqrt{2}} \\ G_a = 0 & \text{on } \partial \Sigma_{\sqrt{2}} \end{cases} , \quad (2.27)$$

where δ_a denotes the Dirac mass at a . From the strong maximum principle $G_a > 0$ on the connected component of $\Sigma_{\sqrt{2}}$ containing a whereas $G_a \equiv 0$ elsewhere. Since $\text{supp}(\chi \eta^k) \subset \Sigma_{\sqrt{2}}$, we have

$$\forall a \in \Sigma_{\sqrt{2}} \quad \chi \eta^k(a) = \int_{\Sigma_{\sqrt{2}}} G_a(x) \Delta_{\Sigma}(\chi \eta^k)(x) dx \quad . \quad (2.28)$$

For $0 \leq s_1 \leq s_2 \leq +\infty$, we denote

$$\mathcal{G}_a^{s_1, s_2} := \{x \in \Sigma_{\sqrt{2}} ; s_1 \leq G_a(x) \leq s_2\} \quad .$$

Using the coarea formula (see [Fe]), we have

$$\int_{\mathcal{G}_a^{s_1, s_2}} |\nabla G_a|^2 = \int_{s_1}^{s_2} ds \int_{G_a^{-1}(s)} |\nabla G_a|(x) d\mathcal{H}^1 \quad . \quad (2.29)$$

Using the fact that for regular values s of G_a , for $x \in G_a^{-1}(s)$ $|\nabla G_a|(x) = -\frac{\partial G_a}{\partial \nu}(x)$ where ν is the outward unit normal to $\mathcal{G}_a^{s, \infty}$ and the fact that

$$\int_{G_a^{-1}(s)} -\frac{\partial G_a}{\partial \nu}(x) d\mathcal{H}^1 = \int_{\mathcal{G}_a^{s, \infty}} \Delta_{\Sigma} G_a = 1 \quad , \quad (2.30)$$

we finally obtain

$$\int_{\mathcal{G}_a^{s_1, s_2}} |\nabla G_a|^2 = s_2 - s_1 \quad . \quad (2.31)$$

Let $\delta > 0$, we deduce from (2.31)

$$\begin{aligned} \frac{1}{\delta} &= \int_{s=1}^{+\infty} s^{-2-\delta} \int_{\mathcal{G}_a^{0, s}} |\nabla G_a|^2 \\ &= \frac{1}{1+\delta} + \frac{1}{1+\delta} \int_1^{+\infty} s^{-1-\delta} \int_{G_a^{-1}(s)} |\nabla G_a| \\ &= \frac{1}{1+\delta} + \frac{1}{1+\delta} \frac{1}{\left(\frac{1}{2} - \frac{\delta}{2}\right)^2} \int_{\mathcal{G}_a^{1, \infty}} |\nabla G_a^{\frac{1}{2}-\frac{\delta}{2}}|^2 \end{aligned} \quad (2.32)$$

Thus, taking $\delta = \frac{1}{2}$, we have

$$\int_{\mathcal{G}_a^{1,\infty}} |\nabla G_a^{\frac{1}{4}}|^2 \leq 48 \quad . \quad (2.33)$$

Let f be a smooth function equal to t on $[0, \frac{1}{2}]$ and equal to $t^{\frac{1}{4}}$ on $[1, +\infty]$. Since $f(G_a) = 0$ on $\partial\Sigma_{\sqrt{2}}$, we can use one by one the arguments of Lemma 2.1 to obtain

$$\begin{aligned} & \left[\int_{\Sigma_{\sqrt{2}}} |f(G_a)|^8 \right]^{\frac{3}{8}} \left[\int_{\Sigma_{\sqrt{2}}} |\nabla f(G_a)|^{\frac{8}{5}} \right]^{\frac{5}{8}} \geq \\ & \int_{\Sigma_{\sqrt{2}}} |f(G_a)|^3 |\nabla f(G_a)| \geq \frac{1}{4} \int_{\Sigma_{\sqrt{2}}} |\nabla f^4(G_a)| \geq \\ & \int_0^{+\infty} \mathcal{H}^1(f^4(G_a)^{-1}(s)) ds \geq K \int_0^{+\infty} ds [x; f^4(G_a)(x) \geq s]^{\frac{1}{2}} \\ & = K \|f(G_a)^4\|_{L^{2,1}(\Sigma_{\sqrt{2}})} \geq K \left[\int_{\Sigma_{\sqrt{2}}} f(G_a)^8 \right]^{\frac{1}{2}} \end{aligned} \quad (2.34)$$

Combining (2.33) and (2.34) we obtain that

$$\int_{\Sigma_{\mathcal{G}_a^{1,+\infty}}} |G_a|^2 \leq K \quad . \quad (2.35)$$

where K has the usual dependence in $d_1(g)$, $d_2(g)$ and $\int_{\Sigma} \Psi^* \omega$. Using the coarea formula again (2.28) becomes

$$\chi \eta^k(a) = \int_0^{+\infty} s ds \int_{G_a^{-1}(s)} \frac{\Delta_{\Sigma}(\chi \eta^k)}{|\nabla G_a|} \quad . \quad (2.36)$$

Since G_a is harmonic aside from a the zeros of ∇G_a are isolated points and then for every $s \in \mathbb{R}_*^+ G_a^{-1}(s)$ is a union of finitely many smooth closed curves aside eventually from isolated points. Therefore, since also $\chi \eta^k$ is smooth, we have that $s \rightarrow \int_{G_a^{-1}(s)} \Delta_{\Sigma}(\chi \eta^k)$ is continuous everywhere and smooth aside from finitely many s corresponding to the values of the finitely many critical points of G_a . Moreover, aside from these points, it's derivative is the function $s \rightarrow - \int_{G_a^{-1}(s)} \frac{\Delta_{\Sigma}(\chi \eta^k)}{|\nabla G_a|}$. For all these reasons we have a BV function without jump points and without Cantor parts in the derivative and the following holds in a distributional sense

$$\frac{d}{ds} \left[\int_{\mathcal{G}_a^{s,+\infty}} \Delta_{\Sigma}(\chi \eta^k) \right] = - \int_{G_a^{-1}(s)} \frac{\Delta_{\Sigma}(\chi \eta^k)}{|\nabla G_a|} \quad \text{in } \mathcal{D}'(\mathbb{R}_*^+) \quad . \quad (2.37)$$

Using a Taylor expansion of the smooth function $\chi \eta^k$ at a it is not difficult to justify the following integration by parts

$$\int_0^{+\infty} s ds \int_{G_a^{-1}(s)} \frac{\Delta_{\Sigma}(\chi \eta^k)}{|\nabla G_a|} = \int_0^{+\infty} \int_{\mathcal{G}_a^{s,+\infty}} \Delta_{\Sigma}(\chi \eta^k) \quad . \quad (2.38)$$

We then have, using also (2.26)

$$\begin{aligned}
 |\chi\eta^k|(a) &\leq \int_0^{+\infty} ds [\mathcal{H}^2(\mathcal{G}_a^{s,+\infty})]^{\frac{1}{2}} \left[\int_{\Sigma_{\sqrt{2}}} |\Delta_{\Sigma}(\chi\eta^k)|^2 \right]^{\frac{1}{2}} \\
 &\leq K \|\nabla J\|_{\infty} \int_0^{+\infty} ds [\mathcal{H}^2(\mathcal{G}_a^{s,+\infty})]^{\frac{1}{2}} \\
 &\leq K \|\nabla J\|_{\infty} \int_0^{+\infty} ds \mathcal{H}^1(G_a^{-1}(s)) \quad ,
 \end{aligned} \tag{2.39}$$

where we have used the fact that, for $s > 0$, $G_a^{-1}(s)$ is contained in the interior of $\Sigma_{\sqrt{2}}$, is the boundary of $\mathcal{G}_a^{s,+\infty}$ and the fact that, $\Sigma_{\sqrt{2}}$ being an area minimizing surface, it inherits the isoperimetric constant of the ambient space (\mathbb{R}^{2p}, g) depending only on $d_1(g)$ and $d_2(g)$. We have, using (2.31)

$$\begin{aligned}
 \int_0^{+\infty} ds \mathcal{H}^1(G_a^{-1}(s)) &= \int_{\mathcal{G}_a^{0,1}} |\nabla G_a| + \int_1^{+\infty} ds \mathcal{H}^1(G_a^{-1}(s)) \\
 &\leq K + \int_1^{+\infty} \left[\int_{G_a^{-1}(s)} |\nabla G_a| \right]^{\frac{1}{2}} \left[\int_{G_a^{-1}(s)} \frac{1}{|\nabla G_a|} \right]^{\frac{1}{2}} \\
 &\leq K + \int_1^{+\infty} \left[\int_{G_a^{-1}(s)} \frac{\partial G_a}{\partial \nu} \right]^{\frac{1}{2}} \left[\int_{G_a^{-1}(s)} \frac{1}{|\nabla G_a|} \right]^{\frac{1}{2}} \\
 &\leq K + \int_1^{+\infty} \frac{1}{s} \left[\int_{G_a^{-1}(s)} \frac{s^2}{|\nabla G_a|} \right]^{\frac{1}{2}} \\
 &\leq K + K \left[\int_{\mathcal{G}_a^{1,+\infty}} G_a^2 \right]^{\frac{1}{2}} \leq K
 \end{aligned} \tag{2.40}$$

where K is controlled by the usual quantities. Combining (2.39) and (2.40) we obtain (1.5) and Theorem 1.1 is proved. \square

A Appendix

Definition A.1 A Riemann surface Σ is said to be finite if each connected component Σ^k of Σ is an open subset of a closed Riemann surface $\tilde{\Sigma}^k$ and $\partial\Sigma^k$ is a non-empty finite union of closed regular curves embedded in $\tilde{\Sigma}^k$.

We have the following classical proposition (see for instance [FK]).

Proposition A.1 Let Σ be a finite Riemann surface, then Σ is hyperbolic (admits a Green function) and for every $\phi \in W^{\frac{1}{2},2}(\partial\Sigma, \mathbb{C})$, there exists a unique $u \in W^{1,2}(\Sigma, \mathbb{C})$ such that

$$\begin{cases} \Delta u = 0 & \text{in } \Sigma \\ u = \phi & \text{on } \partial\Sigma \end{cases}$$

Observe that spaces and equations above are independant of the metric chosen compatible with the complex structure on Σ . We consider the following Hermitian scalar product on $L^2(\Sigma, \mathbb{C})$

$$\langle \omega_1, \omega_2 \rangle := \int_{\Sigma} \omega_1 \wedge * \bar{\omega}_2 \quad , \quad (\text{A.1})$$

and the following antihermitian sesquilinear form on $L^2(\Sigma, \mathbb{C})$

$$(\omega_1, \omega_2) := \int_{\Sigma} \omega_1 \wedge \bar{\omega}_2 \quad . \quad (\text{A.2})$$

(recall that if ξ are local complex coordinates in Σ $*d\xi = id\xi$ and $*d\bar{\xi} = -id\bar{\xi}$). Recall that $H_{\pm}^0(\Sigma)$ denote the sets of holomorphic and antiholomorphic functions in $W^{1,2}(\Sigma, \mathbb{C})$. Let $\partial H_+^0(\Sigma)$ and $\bar{\partial} H_-^0(\Sigma)$ be the sets of exact holomorphic and antiholomorphic 1-forms. Let Γ_k for $k = 1 \cdots q$ be the connected components of Σ . Denote $\mathcal{H}^0(\Sigma)$ the set of harmonic functions in $W^{1,2}(\Sigma, \mathbb{C})$. Let v_i be the solution of

$$\begin{cases} \Delta v_i = 0 & \text{in } \Sigma \\ v_i = \delta_{ik} & \text{on } \Gamma_k \quad \text{for } k = 1 \cdots q \end{cases} \quad (\text{A.3})$$

(δ_{ik} are the Kronecker Symbols). Finally we introduce the following notation

$$V := \text{Vect}_{\mathbb{C}} \{v_1, \dots, v_q\} \quad . \quad (\text{A.4})$$

The following proposition holds.

Proposition A.2 *Let Σ be a finite Riemann surface. Then the following orthogonal decomposition of $d\mathcal{H}^0(\Sigma)$ for $\langle \cdot, \cdot \rangle$ holds*

$$d\mathcal{H}^0(\Sigma) = \partial H_+^0(\Sigma) \oplus \bar{\partial} H_-^0(\Sigma) \oplus dV \quad . \quad (\text{A.5})$$

Remark A.1 Observe that the above decomposition (A.5) corresponds to a Sylvester decomposition of $d\mathcal{H}^0(\Sigma)$ for the Hermitian form $i^{-1}(\cdot, \cdot)$. Precisely on $\partial H_+^0(\Sigma)$ the sesquilinear form $i^{-1}(\cdot, \cdot)$ is definite positive, on $\bar{\partial} H_-^0(\Sigma)$, $i^{-1}(\cdot, \cdot)$ is definite negative and, on dV , (\cdot, \cdot) is identically zero.

Proof of Proposition A.2. First of all we construct a particular basis of the de Rham Group $H^1(\Sigma, \mathbb{R})$ by taking Poincaré-Lefschetz duals of some chosed basis of $H_1(\Sigma, \partial\Sigma)$. Since $\partial\Sigma$ is non-empty and has a finite topology it is homeomorphic (see [Ma]) to the disk D^2 to which $q-1$ disjoint subdisks, D_1, \dots, D_{q-1} , that we may assume to be included in $D_-^2 := D^2 \cap \{(x, y) ; x \leq 0\}$, have been removed, to which $2p$ other disjoint subdisks $d_1, d_2 \cdots d_{2p}$, that we may assume to be included in $D_+^2 := D^2 \cap \{(x, y) ; x \geq 0\}$, have also been removed and to which, finally, p Handels $h_l = S^1 \times [0, 1]$ $l = 1 \cdots p$, have been glued by identifying the two connected components of ∂h_l with respectively d_{2i-1} and d_{2i} . We now chose the following basis for $H_1(\Sigma, \partial\Sigma)$. First for each $l = 1 \cdots p$ we chose γ_l to be ∂d_{2l-1} and $\delta_l \subset D_+^2$ to be a closed curve in $(D_+^2 \setminus \cup_l d_l) \cup \bar{h}_l$

made of the meridian $\{(0, 1)\} \times [0, 1]$ and a curve in $(D_+^2 \setminus \cup_l d_l)$ connecting the two ends of this meridian. We can assume that the δ_l do not intersect each-other. We then complete the families $(\gamma_l)_{l=1\dots p}$ and $(\delta_l)_{l=1\dots p}$ by a collection of $q - 1$ curves η_l in D_-^2 , each curve η_l connecting the segment ∂D_l with the boundary ∂D^2 . $(\gamma_l)_{l=1\dots p}$, $(\delta_l)_{l=1\dots p}$ and $(\eta_l)_{l=1\dots q-1}$ form a basis of $H_1(\Sigma, \partial\Sigma)$. If we add this time the family of circles $(\partial D_l)_{l=1\dots q-1}$ to $(\gamma_l)_{l=1\dots p}$, $(\delta_l)_{l=1\dots p}$ we get a basis of $H_1(\Sigma)$. Consider one curve c taken among the two first types $c = \gamma_l$ or $c = \delta_l$. Since the intersection number of c with the η_l is zero, by the standard construction method (see [BT]), one gets the existence of a representant α_c of the Lefschetz-Poincaré dual of c in $H^1(\Sigma, \partial\Sigma)$ (see the relative to the boundary de Rham cohomology pages 78-79 of [BT], corresponding here also to the compactly supported de Rham cohomology) which is compactly supported in Σ :

$$\exists \alpha_c \in C_0^\infty(\wedge^1 \Sigma) \cap \text{Kerd} \quad \text{s. t. } \forall \phi \in C^\infty(\wedge^1 \Sigma) \int_\Sigma \alpha_c \wedge \phi = \int_c \phi \quad . \quad (\text{A.6})$$

Among the representants of the class given by the Lefschetz-Poincaré we choose the Coulomb Gauge minimizing the following problem

$$\min \left\{ \begin{array}{l} \alpha \in C^\infty(\Sigma) \cap \text{Kerd} \quad \iota_{\partial\Sigma}^* \alpha = 0 \\ \int_\Sigma |d^* \alpha|^2 \\ \forall \phi \in C^\infty(\wedge^1 \Sigma) \int_\Sigma \alpha_c \wedge \phi = \int_c \phi \end{array} \right. \quad . \quad (\text{A.7})$$

where $\iota_{\partial\Sigma}$ is the canonical embedding of $\partial\Sigma$ in $\overline{\Sigma}$. The minimizer α_c^0 solves then

$$\left\{ \begin{array}{l} d\alpha_c^0 = 0 \quad \text{in } \Sigma \\ d^* \alpha_c^0 \quad \text{in } \Sigma \\ \iota_{\partial\Sigma}^* \alpha_c^0 \\ \forall \phi \in C^\infty(\wedge^1 \Sigma) \cap \text{Kerd} \quad \int_\Sigma \alpha_c^0 \wedge \phi = \int_c \phi \end{array} \right. \quad (\text{A.8})$$

(the uniqueness of α_c^0 comes from the following fact : if β solves $d\beta = 0$ in Σ , $d^* \beta = 0$ in Σ , $\iota_{\partial\Sigma}^* \beta = 0$ and $\int_\Sigma \beta \wedge \phi = 0$ for any $\phi \in C^\infty(\wedge^1 \Sigma) \cap \text{Kerd}$ then $\beta = dh$ where h solves $\Delta h = 0$ in Σ and $\frac{\partial h}{\partial \nu} = 0$ which clearly implies that h is constant and therefore that $\beta = 0$). Take now $f \in \mathcal{H}^0(\Sigma)$ and assume that

$$\forall k = 1, \dots, q \quad \int_{\Gamma_k} *df = 0 \quad . \quad (\text{A.9})$$

Then we claim that $*df$ is exact in Σ . For any $c \in \{\gamma_l\} \cup \{\delta_l\}$ we have

$$\int_\Sigma *df \wedge \alpha_c^0 = - \int_\Sigma df \wedge * \alpha_c^0 = \int_\Sigma f \wedge d * \alpha_c^0 - \int_{\partial\Sigma} f * \alpha_c^0 = 0 \quad . \quad (\text{A.10})$$

Combining (A.9) and (A.10) we get that $*df$ is null-cohomologic and is therefore exact which proves the claim. Let then h such that $dh = *f$, we have

$$f = \frac{1}{2}(f + ih) + \frac{1}{2}(f - ih) \in H_+^0(\Sigma) \oplus H_-^0(\Sigma).$$

Thus, the codimension of $dH_+^0(\Sigma) \oplus dH_-^0(\Sigma)$ in $\mathcal{H}^0(\Sigma)$ is at most $q - 1$ (because we have to subtract the relation $0 = \int_{\Sigma} d * df = \sum_{k=1}^q \int_{\Gamma_k} *df$). Let v_j one of the function introduced in (A.3) and let $f \in H_+^0(\Sigma)$, then we have

$$\begin{aligned} \langle dv_j, df \rangle &= \int_{\Sigma} dv_j \wedge *d\bar{f} = \int_{\Sigma} dv_j \wedge *d\Re f - i \int_{\Sigma} dv_j \wedge d\Im f \\ &= \int_{\Sigma} dv_j \wedge d\Im f + i \int_{\Sigma} dv_j \wedge d\Re f = \int_{\Gamma_j} d\Im f + i \int_{\Gamma_j} d\Re f = 0 \quad . \end{aligned} \quad (\text{A.11})$$

Thus, we have that dV is perpendicular to $dH_+^0(\Sigma)$ and a similar argument shows that it is also perpendicular to $dH_-^0(\Sigma)$. Thus $dV \perp (dH_+^0 \oplus dH_-^0)$. It is also straightforward to check that the dimension of dV is $q - 1$. Therefore $d\mathcal{H}^0 = dV \oplus dH_+^0 \oplus dH_-^0$ and Proposition A.2 is proved. \square

Proposition A.3 *Let Σ be a finite Riemann surface whose connected components are denoted by Σ_k , $k = 1 \cdots n$. Let $\psi \in W^{1,2}(\Sigma, \mathbb{C})$, there exists a unique complex valued function $\eta \in W^{1,2}(\Omega, \mathbb{C})$*

$$\left\{ \begin{array}{l} \bar{\partial}\eta = \bar{\partial}\psi \quad \text{in } \Sigma \\ \tilde{\eta} \in H_-^0(\Sigma) \oplus V \quad , \\ \forall \Sigma_k \text{ connected compo. of } \Sigma, \quad \int_{\partial\Sigma_k} \eta = 0 \end{array} \right. \quad (\text{A.12})$$

where $\tilde{\eta}$ is the harmonic extension of the restriction of η to $\partial\Sigma$ inside Σ . Moreover we have

$$\int_{\Sigma} |\nabla\eta|^2 \leq 2 \int_{\Sigma} |\bar{\partial}\psi|^2 \quad . \quad (\text{A.13})$$

Proof of Proposition A.3. Let $\tilde{\psi}$ be the harmonic extension of ψ restricted to $\partial\Sigma$ inside Σ . From Proposition A.2, $d\tilde{\psi}$ admits a unique decomposition $d\tilde{\psi} = d\psi_+ + d\psi_- + d\psi_V$ where $d\psi_{\pm} \in H_{\pm}^0(\Sigma)$ and $d\psi_V \in dV$. We chose η such that $d\eta = d\psi - d\psi_+$ with constant adjusted in such a way that $\forall k = 1 \cdots n \quad \int_{\partial\Sigma_k} \eta = 0$ where $\tilde{\eta}$ is the harmonic extension of η restricted to $\partial\Sigma$ inside Σ . η clearly solves (A.12). The uniqueness is given by the fact that a solution to

$$\left\{ \begin{array}{l} \bar{\partial}\delta = 0 \quad \text{in } \Sigma \\ \tilde{\delta} \in H_-^0(\Sigma) \oplus V \end{array} \right. \quad (\text{A.14})$$

is constant on each connected component of Σ . This is a direct consequence of Proposition A.2 since $\bar{\partial}\delta = 0$ is equivalent to $\bar{\eta} = \eta \in H_+^0(\Sigma)$ and $dH_+^0(\Sigma) \cap dH_-^0(\Sigma) \oplus dV = \{0\}$.

Integration by parts gives

$$\begin{aligned}
 \int_{\Sigma} \left| \frac{\partial \eta}{\partial z} \right|^2 - \left| \frac{\partial \eta}{\partial \bar{z}} \right|^2 dz \wedge d\bar{z} &= \int_{\Sigma} d\eta \wedge d\bar{\eta} \\
 &= \int_{\partial \Sigma} \eta d\bar{\eta} = \int_{\partial \Sigma} \eta_- + \eta_V d(\bar{\eta}_- + \bar{\eta}_V) \\
 &= \int_{\partial \Sigma} \eta_- d\bar{\eta}_- = \int_{\Sigma} d\eta_- \wedge d\bar{\eta}_- \\
 &= - \int_{\Sigma} \left| \frac{\partial \eta_-}{\partial \bar{z}} \right|^2 dz \wedge d\bar{z} \leq 0
 \end{aligned} \tag{A.15}$$

Therefore we have

$$\int_{\Sigma} \left| \frac{\partial \eta}{\partial z} \right|^2 dz \wedge d\bar{z} \leq \int_{\Sigma} \left| \frac{\partial \eta}{\partial \bar{z}} \right|^2 dz \wedge d\bar{z} = \int_{\Sigma} \left| \frac{\partial \psi}{\partial \bar{z}} \right|^2 dz \wedge d\bar{z}$$

and Proposition A.3 follows. \square

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